

Analysis of quantum stochastic differential equations in driven cavity single mode

Lazhar Bougoffa*and Smail Bougouffa†

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Abstract

Various approaches are investigated in order to analyze numerous stochastic differential equations that can be encountered in quantum optics problems, especially, in the case of the driven single cavity mode. The passage to the ordinary coupled differential equations is presented and the treatment of the obtained coupled system is explored. Generalization of the problem to stimulate variable coefficients is discussed and the exact solutions are achieved in explicit forms under suitable conditions on the coefficients.

Keywords: Driven cavity mode, Fokker-Planck equation, master equation, Ito stochastic differential equation, Coupled system, exact solution, homogeneous system, non-homogeneous system.

1 Introduction

It is well known in quantum optics problems, there are quantum fluctuations associated with the states corresponding to classically well-defined electromagnetic fields. The general explanation of fluctuation phenomena needs the density operator. Nevertheless, it is possible to give an other option but equivalent description in terms of distribution functions [1–3]. It has shown that the extended treatment of quantum statistical phenomena by developing the theory of quasi classical distributions is very interesting field of investigation [1]. This is of interest for several motivations. In the begining, the expansion of the quantum theory of radiation to involve nonquantum stochastic effects such as

*Al Imam Mohammad Ibn Saud Islamic University (IMSIU), Faculty of Science, Department of Mathematics, P.O. Box 90950, Riyadh 11623, Saudi Arabia. E-mail address: lbougoffa@imamu.edu.sa

†Department of Physics, Faculty of Science, Taibah University P.O. Box 30002, Madinah 41481, Saudi Arabia E-mail address: sbougouffa@hotmail.com or sbougouffa@taibahu.edu.sa

thermal fluctuations is required. This is an important factor in the theory of partial coherence. In addition, the edge between classical and quantum physics is explained by the use of such distribution. The arch type example being the Wigner distribution [4].

In addition, the investigation of the border between quantum and classical physics is an attractive issue. Nowhere is this better demonstrated than in quantum optics, where we are regularly encountered with the problem of describing fields which are nearly classical but have significant quantum characters. The coherent states are well appropriated to such studies [1–3].

On the other hand, in many quantum optics problems, it is appropriate to illustrate the state of the field in terms of coherent states, rather than with photon number states. This presents some revelations and complications [1–3], the coherent states are not orthogonal and are over complete. In addition, as we shall notice this over completeness permits us to get a helpful diagonal expansion of the density operator in terms of complex matrix elements $P(\alpha)$. This representation can be understood as a quasi-probability distribution function, whose dynamics can under suitable conditions be described in the form of a Fokker-Planck equation [5–10].

Now in quantum optics, almost problems are governed by the master equation, which is an operator type equation that is not easy to solve. Then we have recourse to use some techniques in order to solve these problems. One of them is the use of the probability distribution that can be appeared from density operator. The reason of the representation of the master equation was to attain c-number differential equations that are equivalent to the operator equations, but are more willingly soluble. In particular, the P-representation is used to estimate the normally ordered correlation functions of the field operators. Furthermore, the P-representation forms a correspondence between the quantum and the classical coherent theory.

In this work, we will examine the coherent state representation, P representation, which transforms the master equation into a c-number differential equation called the Fokker-Planck equation for an interesting case, the master equation of a harmonic oscillator in a squeezed thermal reservoir. We then illustrate how the equation can be viewed as a stochastic equation that, for specific initial conditions, can be solved analytically by a direct integration.

The latter part of the work shows the technique of solving the Fokker-Planck equation when direct solutions are not possible. This technique involves stochastic differential equations approach and will be illustrated on two models of typical quantum optics problems: Single cavity mode driven by a classical coherent field and effect of two-photon losses on the driven cavity mode. Thus, the obtained system is nonlinear coupled differential equations, which, in general cannot always be analytically solved. We present some treatments of the obtained equations under some suitable conditions. It may, however, be worthwhile if the physical models can be constructed in such manner that the coupled nonlinear system can either be solved analytically or transformed into another system in which the equations are ordinary and can be decoupled, then solved separately. In this work we will examine various approaches to solve these obtained cou-

pled nonlinear systems and then discuss the possibility to enlarge this approach for the case of variable coefficients. Within this scope this paper begins in Sec. 2 with setting up the master equation a system of a harmonic oscillator in a squeezed thermal reservoir and the corresponding Fokker-Planck equation is obtained. Consequently, the obtained stochastic equation is completely solved in the steady state regime. In Sec. 3 we establish the relationship between the master equation and the stochastic differential equations with two interesting models in quantum optics. Sec. 4 we explore several approaches to solve analytically the obtained equations for some class of initial conditions and the possibility to transform the stochastic equations to a system of nonlinear coupled ordinary equations. In Sec. 5, we summarize and conclude our results.

2 Density Matrix Equation (DME) and Linear Fokker-Planck Equation (LFPE)

In many quantum optics problems, it is constructive to illustrate the state of the field in terms of coherent states, rather than with photon number states. However, in this case some revelations and complications can be presented [2]. Indeed, the coherent states are not orthogonal and are over-complete. On the other hand, as we shall see this over-completeness permits us to get a valuable diagonal expansion of the density operator in terms of the probability distribution function $P(\alpha)$ in coherent state representation. In addition, this representation can be understood as a quasi-probability distribution function, whose dynamics can under suitable conditions be expressed in the form of a Fokker-Planck equation [7]. A number of additional quasi-probability distribution descriptions of the electromagnetic field can be established, including the Wigner function $W(\alpha)$ [4,8] and the Q-function $Q(\alpha)$ [4,9]. These diverse representations, which are called quasiprobability functions, as they are not positive-definite, recover applications in the estimation of correlation functions of the electromagnetic field.

In order to show the efficiency of these representations, we consider a single-mode electromagnetic field in a cavity interacting with a multimode field (reservoir) [11–14, 16], whose the modes are in vacuum thermal states. An understanding of the dynamics of such system is of great exploit in quantum optics as, in principle at least, all problems involving bosonic fields can be represented in terms of the harmonic oscillator. In the interaction picture and the rotating-wave approximation, the general interaction Hamiltonian between cavity mode and the many oscillators reservoir is given by

$$H_1 = \hbar \sum_{\mathbf{k}} \left(g_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} a e^{i(\omega_k - \omega)t} + g_{\mathbf{k}}^* a^{\dagger} b_{\mathbf{k}} e^{-i(\omega_k - \omega)t} \right) \quad (2.1)$$

where a (and a^{\dagger}) are the destruction (and creation) operators of the mode of interest with frequency ω . the operators $b_{\mathbf{k}}$ and $b_{\mathbf{k}}^{\dagger}$ represent modes of reservoir with frequency ω_k , which damp the field and $g_{\mathbf{k}}$ is the coupling strength. We

assume a stimulating wave is added, then the corresponding Hamiltonian can be read as

$$H_2 = i\hbar \left(E e^{-i(\omega_L - \omega)t} a^\dagger - E^* e^{i(\omega_L - \omega)t} a \right) \quad (2.2)$$

where E is the amplitude (Rabi frequency) and ω_L is the angular frequency of the classical coherent field. Then the master equation for the reduced density operator for the field $\rho(t)$ can be achieved [1, 11–14]

$$\begin{aligned} \frac{d}{dt}\rho(t) = & \left[-i\delta a^\dagger a + E a^\dagger - E^* a, \rho \right] \\ & - \frac{\gamma}{2}(N+1) \left([a^\dagger, a\rho(t)] + [\rho(t)a^\dagger, a] \right) \\ & - \frac{\gamma}{2}N \left([a, a^\dagger\rho(t)] + [\rho(t)a, a^\dagger] \right) \\ & - \frac{\gamma}{2}M \left([a\rho(t), a] + [a, \rho(t)a] \right) \\ & - \frac{\gamma}{2}M^* \left([a^\dagger\rho(t), a^\dagger] + [a^\dagger, \rho(t)a^\dagger] \right), \end{aligned} \quad (2.3)$$

where $\delta = \omega - \omega_L$ is the detuning between the cavity mode and the laser frequency and γ is the decay rate of the modes and N is the mean number of photons in the thermal reservoir. The first term in Eq. (2.2) represents decay of the field mode with the rate $\gamma(N+1)$, the second term represents an incoherent pumping of the modes with rate γN . The third and fourth terms represent correlations between photons with the degree $M = |M| \exp(-i\theta)$. The parameter $|M|$ determines the degree of two-photon correlations inside the mode. Therefore, it is referred to as the degree of a single-mode squeezing. θ being the reference phase for the squeezed field which surrounds the cavity. $[\cdot, \cdot]$ represents the commutator operator. It should be pointed out that there is a clear distinction between classical and nonclassical (quantum) regimes for squeezing [15] that involve classical and quantum correlations, respectively. The regimes are determined by the degree of correlations $|M|$. The classical regime for squeezing, often called to as a classical squeezing, is determined by $0 < |M| \leq N$, whereas $N < |M| \leq \sqrt{N(N+1)}$ indicates the nonclassical regime for squeezing, often called to as quantum squeezing. Note that the field with $M = 0$ and $N > 0$ is a thermal field while the field with $M = 0, N = 0$ is the ordinary vacuum field. The parameters M and N can also be expressed in terms of the squeezing parameter r , $M = \sinh r \cosh r$ and $N = \sinh^2 r$.

We will show the main steps of the derivation of the LFPE. Suppose that there exists a time-dependent P distribution $P(\alpha; t)$. In this case, using the P representation for the density operator of the cavity modes

$$\rho = \int d^2\alpha P(\alpha) |\alpha\rangle \langle\alpha|, \quad (2.4)$$

where $d^2\alpha = d\text{Re}(\alpha)d\text{Im}(\alpha)$. With this P representation, the DME (2.2) can be mapped to a FPE

$$\begin{aligned} \frac{\partial}{\partial t} P(\alpha, t) &= \left(\frac{\partial}{\partial \alpha} [(\frac{\gamma}{2} + i\delta)\alpha - E] + \frac{\partial}{\partial \alpha^*} [(\frac{\gamma}{2} - i\delta)\alpha^* - E^*] \right. \\ &\quad \left. + \frac{\gamma}{2} (2N \frac{\partial^2}{\partial \alpha \partial \alpha^*} - M \frac{\partial^2}{\partial \alpha^2} - M^* \frac{\partial^2}{\partial \alpha^2}) \right) P(\alpha, t) \end{aligned} \quad (2.5)$$

The term proportional to N represents thermal noise, the terms proportional to M take place from the non-linearity of the medium, and must be considered as a merely quantum phenomenon. This equation is in the form of a Fokker-Planck equation for the quasi-probability $P(\alpha, t)$ of finding the cavity mode in the coherent state $|\alpha\rangle$ at time t .

In general, The Fokker-Planck equations do not have exact solutions, except for linear cases or one dimensional systems. Despite this, an approximate solution can frequently be found, especially in cases where nonlinear effects do not take place. On the other hand, introducing the variables $\beta = (\frac{\gamma}{2} - i\delta)\alpha - E$ and $\beta^* = (\frac{\gamma}{2} + i\delta)\alpha - E^*$, the LFPE (2.5) can be written in the following form

$$\begin{aligned} \frac{\partial}{\partial t} P(\beta, \beta^*, t) &= \left\{ (\frac{\gamma}{2} + i\delta) \frac{\partial}{\partial \beta} \beta + (\frac{\gamma}{2} - i\delta) \frac{\partial}{\partial \beta^*} \beta^* + \frac{\gamma}{2} [2N (\frac{\gamma^2}{4} - \delta^2) \frac{\partial^2}{\partial \beta \partial \beta^*} \right. \\ &\quad \left. - M (\frac{\gamma}{2} + i\delta)^2 \frac{\partial^2}{\partial \beta^2} - M^* (\frac{\gamma}{2} - i\delta)^2 \frac{\partial^2}{\partial \beta^2}] \right\} P(\alpha, t) \end{aligned} \quad (2.6)$$

Equation (2.6) can be rewritten by introducing $\beta = x_1 + ix_2$ and using a vector notation $\mathbf{x} = (x_1, x_2)^T$, $\nabla = (\partial x_1, \partial x_2)^T$.

$$\frac{\partial}{\partial t} P(\mathbf{x}, t) = \left(-\nabla \cdot \mathbf{A} + \frac{1}{2} \sum_{j,l=1}^2 \frac{\partial^2}{\partial x_j \partial x_l} D_{j,l} \right) P(\mathbf{x}, t) \quad (2.7)$$

where $\mathbf{A} = -\frac{\gamma}{2}\mathbf{x}$ is the drift vector that represents the motion of $P(\mathbf{x}, t)$ to origin and $D(2 \times 2)$ is the diffusion matrix that determines the circular shape of $P(\mathbf{x}, t)$ and defined as

$$D = \frac{\gamma}{2} \begin{pmatrix} a & b \\ b & c \end{pmatrix}. \quad (2.8)$$

where

$$\begin{aligned} a &= \frac{1}{4} \left((2N - M - M^*) (\frac{\gamma^2}{4} - \delta^2) + i\gamma\delta(M^* - M) \right), \\ c &= \frac{1}{4} \left((2N + M + M^*) (\frac{\gamma^2}{4} - \delta^2) - i\gamma\delta(M^* - M) \right), \\ b &= \frac{1}{4} \left(M^* (\frac{\gamma^2}{4} - \delta^2 - i\gamma\delta) - M (\frac{\gamma^2}{4} - \delta^2 + i\gamma\delta) \right). \end{aligned} \quad (2.9)$$

In fact the condition for positivity of the diffusion matrix is $|a + c| \geq |b|$, and this is not always satisfied. At first look, this is a contradiction, the P-function is considered as an adequate probability function, and however the consequent

equation of motion does not have a positive diffusion matrix, and for this reason does not always have solutions which are positive. The response to this problem essentially lies in the limited variety of initial conditions [18]. To keep away from this problem we will investigate the solution of this LFPE in the stationary regime. The stationary regime constitutes an interesting regime for many physical systems interacting with reservoirs. To explore the solution in this regime, the LFPE (2.7) can be written as

$$\partial x_j \left(-A_j + \frac{1}{2} \sum_{l=1}^2 \frac{\partial}{\partial x_l} D_{jl} \right) P^s(\mathbf{x}) = 0 \quad (2.10)$$

Seek the Ansatz for P^s satisfying the stationary condition:

$$2A_j P^s - P^s \sum_{l=1}^2 \frac{\partial D_{jl}}{\partial x_l} = \sum_{l=1}^2 D_{jl} \frac{\partial P^s}{\partial x_l} \quad (2.11)$$

We assume that $P^s \neq 0$ and dividing by P^s , then

$$\sum_{l=1}^2 D_{jl} \frac{\partial}{\partial x_l} \ln(P^s) = 2A_j - \sum_{l=1}^2 \frac{\partial D_{jl}}{\partial x_l} \quad (2.12)$$

Suppose $P^s(\mathbf{x}) = e^{-f(\mathbf{x})}/B$ and D invertible, where B is a constant of normalization. Then we deduce that

$$-\frac{\partial f}{\partial x_k} = 2 \sum_{l=1}^2 D_{kj}^{-1} \left[A_j - \frac{1}{2} \sum_{l=1}^2 \frac{\partial D_{jl}}{\partial x_l} \right], \quad (2.13)$$

the second member of this equation is well known from the problem and can be noted as

$$F_k(\mathbf{x}) \equiv 2 \sum_{j=1}^2 D_{kj}^{-1} \left[A_j - \frac{1}{2} \sum_{l=1}^2 \frac{\partial D_{jl}}{\partial x_l} \right], \quad (2.14)$$

then in abbreviation, $\mathbf{F} = -\nabla f$. We assume that the potential conditions are satisfied, i.e.;

$$-\frac{\partial^2 f}{\partial x_k \partial x_l} = \frac{\partial F_l}{\partial x_k} = \frac{\partial F_k}{\partial x_l} = -\frac{\partial^2 f}{\partial x_l \partial x_k}, \quad (2.15)$$

By integration over an arbitrary path, we can obtain the function f

$$f(x) = - \int^{\mathbf{x}} \mathbf{F} \cdot d\mathbf{r} \quad (2.16)$$

For the case where $M = 0, \delta = 0$ and $E = 0$, thus $\mathbf{A} = -\frac{\gamma}{2}\mathbf{x}$ and $D = \gamma N \text{diag}(1, 1)$ we find,

$$D^{-1} = \frac{1}{\gamma N} \text{diag}(1, 1). \quad (2.17)$$

Thus

$$\begin{aligned} f(\mathbf{x}) &= \frac{\gamma}{\gamma N} \left[\int^{x_1} x_1 dx_1 + \int^{x_2} x_2 dx_2 \right] \\ &= \frac{x_1^2 + x_2^2}{2N} \end{aligned} \quad (2.18)$$

The stationary state distribution is a Gaussian with zero mean and variance N ,

$$P(\alpha) = \frac{1}{\pi N} e^{-\frac{|\alpha|^2}{N}}, \quad (2.19)$$

which is the p-representation of a thermal state. Then, if we start, for example, in a squeezed state, we will end up in a thermal state that has lost all information about the initial state.

3 Ito's Stochastic differential equation

In the previous case of initial coherent state, we have pointed out that the multidimensional Fokker-Planck equation can be analytically solved in some special cases. In general, the Fokker-Planck equations are not linear and do not admit direct solutions, thus we have recourse to employ other techniques. One of these techniques is the use of the stochastic differential equations (SDE) approach. This procedure is founded on the fact that for a Fokker-Planck equation with positive diffusion matrix, a set of equivalent stochastic differential equations can be existed. The positive defined diffusion matrix $D_{ij}(\mathbf{x}(t), t)$ can always be factorized into the form

$$D_{ij}(\mathbf{x}(t), t) = \sum_k g_{ik}(\mathbf{x}(t), t) g_{kj}^\dagger(\mathbf{x}(t), t). \quad (3.1)$$

In general, the Itô stochastic differential equation (SDE) [19] can be formulated as

$$dx_i = h_i(\mathbf{x}(t), t)dt + \sum_j g_{ij}(\mathbf{x}(t), t)\xi_j(t)dt \quad (3.2)$$

where $\mathbf{x} = \{x_i | 1 \leq i \leq n\}$ is the set of unknowns, the h_i and g_{ij} are arbitrary functions and $\xi_i(t)$ are real independent Gaussian white noise terms with zero mean value $\overline{\xi_i(t)} = 0$ and delta- δ correlated in time

$$\overline{\xi_i(t)\xi_j(t')} = \delta_{ij}\delta(t - t') \quad (3.3)$$

consequently to the Itô's lemma [19], the set of stochastic differential equations, which is associated to the general form of FPE Eq.(2.5), can be read as

$$dy = \sum_i \frac{\partial y}{\partial x_i} \left(h_i dt + \sum_j g_{ij} \xi_j(t) dt \right) + \frac{1}{2} \sum_{k,l,m} \frac{\partial^2 y}{\partial x_k \partial x_l} g_{km} g_{lm} dt + \frac{\partial y}{\partial t} dt \quad (3.4)$$

where $y = y(x_k, t)$ is a smooth function of the unknown variables x_k . In general the stochastic differential equations can be treated by numerical simulation techniques or by analytic methods if they are linear, but in the case where they are non linear case the situation will become more complicated and needs to recourse to investigate new approaches. In the following, we will consider two special cases that can be generated from the problem of quantum optics. The first one was investigated in the preceding section and is concerned with the single cavity mode interacting with its thermal thermal squeezed environment. The second one is related to the effect of two-photon losses on the driven cavity mode.

3.1 Single cavity mode driven by a classical coherent field

In order to introduce the Ito stochastic differential equations as another alternative to be used to solve some problems faced in quantum optics, these problems engender many interesting physical phenomena. We start by investigating the previous problem of section (2), the corresponding Itô's stochastic differential equations can be obtained

$$\begin{aligned} d\alpha &= \left(-\left(\frac{1}{2}\gamma + i\delta\right)\alpha + E \right) dt + d\eta(t) \\ d\alpha^* &= \left(-\left(\frac{1}{2}\gamma - i\delta\right)\alpha^* + E^* \right) dt + d\eta^*(t) \end{aligned} \quad (3.5)$$

where

$$\begin{aligned} d\eta(t)^2 &= -\gamma M^* dt \\ d\eta^*(t)^2 &= -\gamma M dt \\ d\eta(t)d\eta^*(t) &= \gamma N dt \end{aligned} \quad (3.6)$$

In the case of vacuum reservoir ($N = M = 0$), these equations of motions for α and α^* are not coupled, thus they can be solved exactly by an easy integration, obtaining the solution

$$\alpha(t) = \alpha(0)e^{-(\frac{1}{2}\gamma + i\delta)t} + E \int_0^t e^{-(\frac{1}{2}\gamma + i\delta)(t-t')} dt' \quad (3.7)$$

and the stationary solution of Eq. (3.7) can be also obtained

$$\alpha^s = \frac{E}{\frac{1}{2}\gamma + i\delta} \quad (3.8)$$

Consequently, the coherently driven and damped cavity conserves the coherence, so that the cavity field is all the time in the coherent state. This continuation of coherence under damping is one of the notable possessions of coherent states, which have made them a very common and essential characteristic of laser physics. Nevertheless, the coherence is not conserved if a nonlinear cavity

damping is incorporated. We will point up it in the next example, where the two-photon damping of the cavity mode is included.

In the absence of the monochromatic wave $E = 0$ and exact resonance $\delta = 0$, the equation can be simplified using the quadrature phases, $u = \frac{1}{2}(\alpha + \alpha^*)$, $v = \frac{1}{2i}(\alpha - \alpha^*)$

$$\begin{aligned} du &= -\frac{\gamma}{2}u dt + dW_u(t) \\ dv &= -\frac{\gamma}{2}v dt + dW_v(t) \end{aligned} \quad (3.9)$$

where

$$\begin{aligned} dW_u(t)^2 &= \frac{\gamma}{2}(N + M)dt \\ dW_v(t)^2 &= \frac{\gamma}{2}(N - M)dt \\ dW_u(t)dW_v(t) &= 0 \end{aligned} \quad (3.10)$$

The solutions for these quadrature phase equations are straightforward given by

$$\begin{aligned} u(t) &= u(0)e^{-\frac{\gamma}{2}t} + \int_0^t e^{-\frac{\gamma}{2}(t-t')} dW_u(t'), \\ v(t) &= v(0)e^{-\frac{\gamma}{2}t} + \int_0^t e^{-\frac{\gamma}{2}(t-t')} dW_v(t'), \end{aligned} \quad (3.11)$$

and the variances can be easily evaluated. In general, the density matrix equation (DME) gave for p-function a linear Fokker-Planck equation that its diffusion matrix was not always positive definite. This fact was motivated many researchers to introduce a class of generalized P-representations, by expanding in non-diagonal coherent state projection operators [20]. On the other hand, it can be used to explore the case of quantum noise. This point is beyond our investigation in this paper.

3.2 Two-photon losses effect on the driven cavity mode

Assume that the cavity mode, considered in the previous example, is in addition damped by two-photon losses [21, 22], e.g., due to a two-photon absorption. Then, its corresponding interaction Hamiltonian can be read in the interaction picture as

$$H_3 = \hbar \sum_{\mathbf{k}} \left(G_{\mathbf{k}} b_{\mathbf{k}}^\dagger a^2 + G_{\mathbf{k}}^* (a^\dagger)^2 b_{\mathbf{k}} \right). \quad (3.12)$$

Assume the heat bath is at zero temperature, the master equation of the system can be written as

$$\begin{aligned} \frac{\partial}{\partial t} \rho &= \left[-i\delta a^\dagger a + E a^\dagger - E^* a, \rho \right] - \frac{1}{2} \kappa \left([a^{\dagger 2}, a^2 \rho] + [\rho a^{\dagger 2}, a^2] \right) \\ &\quad - \frac{\gamma}{2} \left([a^\dagger, a \rho(t)] + [\rho(t) a^\dagger, a] \right) \end{aligned} \quad (3.13)$$

where κ is the photon-photon interaction term (the two-photon loss coefficient). The obtained density matrix can be converted to Fokker-Planck equation using the previous treatment. However, the diffusion matrix of FPE is not always positive defined. Furthermore, as the stochastic processes are independent, the corresponding stochastic differential equations for α and α^* are not complex conjugate. To remedy this problem we will use the positive P representation. In this situation, the stochastic differential equations can be obtained only by using the positive P-representation; the usual P-representation does not work [3]. The positive P-representation is defined as follows

$$\rho = \int_{\mathcal{D}} \Lambda(\alpha, \beta) P(\alpha, \beta) d^2\alpha d^2\beta, \quad (3.14)$$

where

$$\Lambda(\alpha, \beta) = \frac{|\alpha\rangle \langle \beta^*|}{\langle \beta^* | \alpha \rangle}, \quad (3.15)$$

and (α, β) vary independently over the whole complex plane \mathcal{D} . The projection operator $\Lambda(\alpha, \beta)$ is analytic in (α, β) .

Subsequent the standard method, using the positive P representation, we convert the master equation into a FPE, then, it can be written in the interaction picture as

$$\begin{aligned} \frac{\partial}{\partial \alpha} P(\alpha, \beta, t) = & \left(\frac{\partial}{\partial \alpha} \left(\left(\frac{1}{2}\gamma + i\delta \right) \alpha + \kappa \alpha^2 \beta^* - E \right) \right. \\ & + \frac{\partial}{\partial \beta^*} \left(\left(\frac{1}{2}\gamma - i\delta \right) \alpha^* + \kappa \beta^{*2} \alpha - E^* \right) \\ & \left. + \frac{1}{2} \frac{\partial^2}{\partial \alpha^2} (-\kappa \alpha^2) + \frac{1}{2} \frac{\partial^2}{\partial \beta^{*2}} (-\kappa \beta^{*2}) \right) P(\alpha, \beta, t). \end{aligned} \quad (3.16)$$

The associated Itô's stochastic differential equations can be obtained

$$\frac{d\alpha}{dt} = a\alpha - \kappa \alpha^2 \beta^* + E + g_{11} \xi_1(t) \quad (3.17)$$

$$\frac{d\beta^*}{dt} = a^* \beta^* - \kappa \beta^{*2} \alpha + E^* + g_{22} \xi_2(t) \quad (3.18)$$

where $a = -(\frac{1}{2}\gamma + i\delta)$ and $\xi_i(t)$ are independent Gaussian noise terms with zero means that satisfying the nonzero correlations Eq.(3.3). The coefficients g_{11} and g_{22} are the diagonal matrix elements of the matrix \mathbf{g} that can be deduced from the diffusion matrix $\mathbf{D} = \mathbf{g}\mathbf{g}^T = \text{diag}[-\kappa\alpha^2, -\kappa\beta^{*2}]$. Then, $g_{11} = i\sqrt{\kappa}\alpha$ and $g_{22} = -i\sqrt{\kappa}\beta^*$.

The equations (4.5, 4.6) constitute a coupled no-linear differential equations of first order. In general, this system is solved numerically or transformed to high order separable differential equations, which also can be treated numerically. In the following, we will investigate an analytic treatment that can be used to solve this nonlinear coupled system.

4 Analytic Method for solving the two nonlinear coupled differential equations

As we have seen in the previous example that the use of the positive P-representation can yield coupled ordinary stochastic differential equations. Then a separation procedure and an analytic treatment of the separated system are required. In the beginning, we will discuss the steady state of the previous coupled system.

4.1 Stationary deterministic solutions

The deterministic stationary solutions of the system (4.5, 4.6) can be obtained by neglecting the noises terms and assuming the time derivatives equal zero. Then

$$\alpha^s = -\frac{a^* Z_0 + E^*}{\kappa Z_0^2} \quad (4.1)$$

$$\beta^{*s} = Z_0 \quad (4.2)$$

where Z_0 is a root of the third order equation

$$\kappa E Z_0^3 + a^*(a - a^*)Z_0^2 + E^*(a - 2a^*)Z_0 - E^{*2} = 0. \quad (4.3)$$

In general, these solutions are not stable except for some special cases. In particular, if the classical coherent field is ignored $E = 0$, then the stationary solution is $\alpha = \beta = 0$ and this solution is stable. Further, in the case of $\gamma = 0$ and $\delta = 0$; i.e., the cavity mode is in resonance with the classical coherent field and the vacuum fluctuations are negligible, then the stationary solution is $\alpha^s = \beta^{*s} = (\frac{E}{\kappa})^{\frac{1}{3}}$. For $E > 0$, this solution is stable. On the other hand, the solutions $\alpha^s = -\beta^{*s} = i^{\frac{2}{3}}(\frac{E}{\kappa})^{\frac{1}{3}}$ are unstable solutions for $E > 0$. However, it is this sort of rearrangement of the deterministic nonlinear dynamics in the extended phase space which can direct to irregular solutions (unstable trajectories) in numerical treatments of the stochastic differential equations when the quantum noises are large.

4.2 Linear noise approximation

In the case of small noise terms, the linear noise approximation can be used, in which the fluctuations are linearized around the steady state solution. on the other hand, we should be conscious that the obtained results will be appropriate just in the framework of this approximation. However, except the problem of passage from one stationary state to another in bistable systems, this is mainly what is obvious well.

Let's consider now the previous coupled system (4.5, 4.6). It is clear that this equation does not include any very apparent small noise factor. Nevertheless, a large driving field regime can be attained by assuming

$$\kappa = \frac{b}{E^2}, \quad \alpha = xE, \quad \beta^* = yE, \quad E^* = E \quad (4.4)$$

then

$$\frac{dx}{dt} = ax - bx^2y + 1 + i\frac{\sqrt{b}}{E}x\xi_1(t), \quad (4.5)$$

$$\frac{dy}{dt} = a^*y - by^2x + 1 - i\frac{\sqrt{b}}{E}y\xi_2(t), \quad (4.6)$$

This clearly shows that in the regime of large driving field, the small linearization approximation can be applied.

4.3 Conversion to coupled ordinary equations

In order to investigate the deterministic solution of the stochastic equations, it is interesting to explore some real physical problem where the stochastic differential equation can be converted to an ordinary coupled differential equations. In the framework of large driving field, the noise terms can be ignored. On the other hand, one of the main problem of mathematics [24] appears when a , b are analytic functions and are added to the original system. Now the new problem, incorporating the above assumptions, is generated by a coupled differential equations,

$$\begin{cases} \frac{dx}{dt} = a(t)x + b(t)x^2y + f(t), \\ \frac{dy}{dt} = c(t)y + d(t)y^2x + g(t). \end{cases} \quad (4.7)$$

A question which arises naturally is under what conditions on the functions $a(t)$, $b(t)$, $c(t)$ and $d(t)$ does the given system have an explicit solution?

In this section, we will present a direct approach to solve the general coupled model by considering the following two important cases.

4.3.1 The homogeneous coupled system

First of all, we begin our approach by considering the following homogeneous coupled system

$$\begin{cases} \frac{dx}{dt} = a(t)x + b(t)x^2y, \\ \frac{dy}{dt} = c(t)y + d(t)y^2x. \end{cases} \quad (4.8)$$

Multiplying both sides of the first and second equation of system (2.3) by y and x , respectively, we get

$$y\frac{dx}{dt} = a(t)xy + b(t)x^2y^2 \quad (4.9)$$

and

$$x\frac{dy}{dt} = c(t)xy + d(t)x^2y^2. \quad (4.10)$$

Adding Eq. (4.9) and Eq. (4.10) together, we obtain

$$y\frac{dx}{dt} + x\frac{dy}{dt} = (a(t) + c(t))xy + (b(t) + d(t))x^2y^2. \quad (4.11)$$

Now let

$$z = xy. \quad (4.12)$$

Hence, Eq. (4.11) becomes

$$\frac{dz}{dt} = A(t)z + B(t)z^2, \quad (4.13)$$

where $A(t) = a(t) + c(t)$ and $B(t) = b(t) + d(t)$, which is a Bernoulli differential equation. The substitution that is needed to solve this Bernoulli equation is

$$z = \frac{1}{u}. \quad (4.14)$$

A set of solutions to Eq. (4.13) is then given by

$$z = \frac{e^{\int A(t)dt}}{-\int B(t)e^{\int A(t)dt}dt + \alpha}, \quad (4.15)$$

where α is a constant of integration.

Inserting Eq. (4.12) into system (4.8) to get

$$\frac{dx}{dt} = (a(t) + b(t)z(t))x \quad (4.16)$$

and

$$\frac{dy}{dt} = (c(t) + d(t)z(t))y. \quad (4.17)$$

Consequently,

$$x(t) = \beta e^{\int (a(t) + b(t)z(t))dt} \quad (4.18)$$

and

$$y(t) = \gamma e^{\int (c(t) + d(t)z(t))dt}, \quad (4.19)$$

where β and γ are two constants of integration, and the function z is given by Eq. (4.15).

Thus we have proved the following result on the separation of this system.

Lemma 1 *The two equations of system (4.7) are decoupled and solvable separately, and the solutions are given by Eq. (4.18) and Eq. (4.19).*

4.3.2 The non-homogeneous coupled system

The system (4.7) can be transformed into another system in which the equations are decoupled and transformed into one equation. This separability can be obtained if

$$y(t) = \varphi(t)x(t), \quad (4.20)$$

where $\varphi(t)$ is unknown function.

The substitution of (4.20) into system (4.7) gives a system which we write as:

$$\begin{cases} \frac{dx}{dt} = a(t)x + b(t)\varphi(t)x^3 + f(t), \\ \frac{dx}{dt} = \frac{c(t)\varphi(t) - \varphi'(t)}{\varphi(t)}x + d(t)\varphi(t)x^3 + \frac{g(t)}{\varphi(t)}. \end{cases} \quad (4.21)$$

Equating coefficients of like terms of system (4.21), we get

$$\frac{c(t)\varphi(t) - \varphi'(t)}{\varphi(t)} = a(t), \quad (4.22)$$

$$b(t) = d(t) \quad (4.23)$$

and

$$\varphi(t) = \frac{g(t)}{f(t)}. \quad (4.24)$$

Eq. (4.22) gives

$$\varphi(t) = \lambda e^{\int (c(t) - a(t)) dt}, \quad (4.25)$$

where λ is a constant.

Thus we have proved the following result on the separation of this system.

Lemma 2 *The system (4.7) can be reduced to the Abel equation of the first kind [23]*

$$\frac{dx}{dt} = f_0(t) + f_1(t)x + f_2(t)x^3, \quad (4.26)$$

where $f_0 = f$, $f_1 = a$ and $f_2 = b\varphi$, such that

$$y = \frac{g(t)}{f(t)}x \quad (4.27)$$

if and only if the following conditions

$$\frac{g(t)}{f(t)} = \lambda e^{\int (c(t) - a(t)) dt} \quad (4.28)$$

and Eq.(4.23) are satisfied.

For the previous problem, we have $b = d$, $a = c^* = -(\frac{\gamma}{2} + i\delta)$ and $f = g = 1$, then $y = x$ and the reduced equation takes the form

$$\frac{dx}{dt} = 1 + ax + bx^3, \quad (4.29)$$

where $\lambda = e^{-\int 2i\delta dt}$.

5 Conclusion

We have explored some techniques to analysis the quantum stochastic differential equations, which are generated in the case of driven single cavity mode in different reservoirs. One of the mean results resides essentially in the conversion of the stochastic equations to ordinary differential equations. Generalization coverage of the nonlinear coupled differential equation is presented. These results stimulate a sequence of questions of mathematical as well as physical consequence. Ideally one would like to be capable of predicting the behavior

of paths for any set of initial conditions and parameter values. This is very much an interesting question in general. The quantum optics models offer an important source in the nonlinear aspects, and have motivated the development of techniques to examine more and more difficult and higher dimensional models.

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